DOCUMENT RESUME

ED 232 874 SE 042 649

AUTHOR Edwards, Allen
TITLE Estimation for Numeracy. Numeracy Project.
INSTITUTION Administrative Coll. of Papua New Guinea, Boroko.
REPORT NO ANR-23
PUB DATE 83
NOTE 40p.; Document contents are of marginal legibility, best copy available.
PUB TYPE Reports - Descriptive (141)

EDRS PRICE MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS Addition; *Arithmetic; *Computer Programs; Division; Elementary School Mathematics; Elementary Secondary Education; *Estimation (Mathematics); Foreign Countries; *Mathematics Education; *Mathematics Instruction; Microcomputers; Multiplication; Secondary Education; Secondary School Mathematics; Subtraction; *Teaching Methods
IDENTIFIERS *New Guinea

ABSTRACT

A program for teaching "number sense" is suggested based on experiences in teaching numeracy to Papua New Guinean adults. Number sense starts out with some sort of "feel" about the size of a number and extends upwards to include any such mental arithmetical skills as can reasonably be incorporated in an appropriate program. What number sense does not include is anything requiring paper/pencil or a calculator. With the exclusion of problem-solving, the major part of number sense consists of the ability to estimate or the ability to obtain approximate answers for problems already stated in numerical terms (arithmetic estimation). Appropriate methods of estimating the total of a column of numbers and the difference and mean of any two numbers are suggested. In addition, methods for multiplication and division are suggested. In order to compare different methods of estimating related to these mathematical processes, a computer program written for the Sinclair ZX Spectrum microputer was employed. Notes on the program and complete program listing are included. (JN)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

U.S. DEPARTMENT OF EDUCATION
NATIONAL INSTITUTE OF EDUCATION
EDUCATIONAL RESOURCES INFORMATION
CENTER (ERIC)
✓ This document has been reproduced as received from the person or organization originating it.
☐ Minor changes have been made to improve reproduction quality.
• Points of view or opinions stated in this document do not necessarily represent official NIE position or policy.

ED232874

ESTIMATION for NUMERACY

Allen Edwards

"PERMISSION TO REPRODUCE THIS MATERIAL IN MICROFICHE ONLY HAS BEEN GRANTED BY
Allen Edwards
TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

SE042649

NUMERACY PROJECT

ADMINISTRATIVE COLLEGE OF PAPUA NEW GUINEA

ANR 23

ESTIMATION FOR NUMERACY

by Allen Edwards MA PhD

EXTRACT

The author draws on his experience in teaching numeracy to Papua New Guinean adults to suggest a programme for teaching "number sense" anywhere. He believes the methods of estimation which he recommends should be considered in the context of formal school education alongside the introduction of the hand-held calculator.

He suggests appropriate methods of estimating the total of a column of numbers and the difference and mean of any two numbers. Before recommending methods for multiplication and division he assesses the merits of a variety of alternative processes using a micro-computer program written by John Edwards to determine the probable accuracy of the estimates.

KEYWORDS: Education, Adult, Mathematics, Numeracy, Mental, Arithmetic, Estimation, Approximation, Micro-computer, Calculator, Computation.

ADMINISTRATIVE COLLEGE OF PAPUA NEW GUINEA

P O Box 1216 Boroko

© Allen Edwards 1983

Computer Program:-
© John Edwards 1983

ANR 23

ESTIMATION FOR NUMERACY

		Page
1.	Number Sense	1
2.	Methods of Estimation	5
	2.1 Addition	5
	2.2 Subtraction	7
	2.3 Means	7
	2.4 Multiplication	8
	2.5 Division	16
3.	Positioning the Decimal Point.	26
4.	Relevant Publications	27
5.	Conclusions	31
6.	Notes on the Micro-computer Program	32
7	Acknowledgements	35
	References	35

1. "Number Sense"

I have argued elsewhere (1) that any teaching of numeracy today should start from the assumption that a hand-held electronic calculator will normally be available whenever it is needed. It follows that teaching the skills necessary to use a calculator should certainly form part of any numeracy programme. What remains very much in doubt however is exactly what else needs to be taught.

My own recent experience is limited to Papua New Guinea where the majority even of those who have been to school can do little more than count and add simple numbers. The reasons for this, in a country with hardly any indigenous arithmetical concepts at all, do not need to be stressed. What would be interesting to know however is just how they compare with say the least numerate 10% in the developed "North".

If there is any significant correlation between educated Papua New Guineans and the least numerate "Northerners", then it may be that research based on Papua New Guinea will indeed be of some assistance to those concerned with the innumerate minority in the North. For reasons I shall suggest later I believe this particular piece of research could also have relevance within the ordinary school curriculum and so be of concern to everyone.

It is not easy to determine what are the next most useful numerical concepts and skills after that of being able to use a calculator. Clearly the ability to count and to add simple numbers come near the top of the list, but not far below them would be what I would describe as "number sense". This "number sense" starts with some sort of "feel" about the size of a number and it then extends upwards to include any such mental arithmetical skills as can reasonably be incorporated in an appropriate programme. What "number sense" does <u>not</u>, to my mind, include is anything that requires either paper and pencil or a calculator. The Cockcroft Report (2) refers to "an at-homeness with numbers" (page 11) and "the feeling for number which permits sensible estimation and approximation" (page 10).

One important aspect of "number sense" which I shall mention and then ignore is very simple "problem solving", that is a sufficient understanding of the concepts of + − x ÷ % proportion etc. to enable them to be applied in any "real life" situation. Nobody would doubt the importance of this, but two considerations

lead me to omit further reference to it from this paper.

First, the teaching of such "problem-solving" is only likely to be successful if the teaching is long term, that is for children in school or for adults with enough motivation to teach themselves. Secondly, it seems to be generally acknowledged (2, page 24) that in"real work" situations people only require to carry out a very limited number of mathematical processes. The average worker does not need this ability to conceptualise in his work situation: someone else has usually done it for him.

With the exclusion of problem-solving then I would say that the major part of "number sense" consists of the ability to estimate. Estimation is of two kinds. What we may call "practical" estimation and "arithmetical" estimation.

Practical estimation consists of measuring or otherwise obtaining an answer in some non-arithmetical way and then being able to express the result in some intelligible form generally involving an explicit or implied range for the possible answer. No attempt will be made here to identify these non-arithmetical skills, and the expression of the result in these cases is similar to those used in arithmetical estimation.

In the rest of this paper therefore we will use the term "estimate" to refer to the use of arithmetical skills to obtain approximate answers for problems already stated in numerical terms. Ideally these skills will only involve processes that can be carried out mentally but in the early stages a minimal amount of actual writing may have to be allowed.

Before proceeding to identify and discuss alternative procedures for estimating I would suggest a reason why I believe these skills to be of importance for school syllabuses in the developed North as well as for non-formal education in the Third World.

Most of us practised our mental arithmetic through frequent use of algorithms such as long multiplication and division: dividing a five figure number by a two figure number for example involves about fifteen mental arithmetic processes. Such complicated algorithms are no longer relevant for they can be done so much better with a calculator. There is therefore a real danger that once the calculator is allowed into the classroom these processes will no long be practised, and with them will go also the essential practice of mental arithmetic.

The only way to retain the mental arithmetic practice while at the same time increasing the "real life" nature of the problems set, would seem to me to be to make estimation an essential part of arithmetical teaching from the very earliest years.

Estimation involves mental arithmetic and enables "real" problems to be "solved". The reason why it has been paid little more than lip-service in the past is, I believe, a simple pedagogical one. If there is no exact answer, how can you tell if anyone is right or wrong? And how can you teach mathematics without right or wrong answers?

This problem can be resolved by defining precise methods of estimation which are to be used, and this is just what this paper is trying to do. Whenever restrictions are placed on the method of estimating to be used however it will of course have to be pointed out that in real life a whole host of different methods may be "correct". But it is easy enough to explain to a student that at this stage a particular procedure is being practised and so <u>must</u> here be used even if other and better methods are known. In non-numerate cultures this is less likely to be a problem, but here it is even more necessary to develop one process of estimating at a time to avoid the confusion that would undoubtedly otherwise ensue.

For any who retain doubts about the importance of estimation and mental arithmetic in mathematical education the following quotations from the Cockcroft Report (2) may be convincing.

In an investigation into the mathematical needs of adults " ..... some felt that there should always be an exact answer to questions involving mathematics and so found themselves in difficulties when it became necessary to approximate or round off a result. 'I get lost on long sums and never know what to do with the leftovers'. 'My mind boggles at the arithmetic in estimation'." (page 8)

"Some who had tried to use a calculator had been discouraged by the large number of figures which had appeared after the decimal point" (page 68).

"Industry and commerce rely extensively on the ability to estimate" (page 22).

"We believe the decline of mental and oral work within the mathematics classrooms represents a failure to recognise the central place which working 'done in the

head' occupies throughout mathematics" (page 75).

"It is therefore essential that children should be helped to attain a secure and rapid recall of addition facts up to 10 + 10 and the related subtraction facts, and of multiplication facts up to 10 x 10 and the related division facts" (page 87).

Estimation is "an area in which further study is required" (page 77).

Finally I must add a quotation from a report by John Hayter (3, page 39) which originally challenged me to try to make sense of the jumble of estimation methods which most of us commonly use.

"In various parts of the world discussions on basic skills in mathematics frequently lead to the listing of 'the ability to estimate' amongst the essential skills .....

"General approval appears often to have been sufficient as a case for inclusion, without I feel a close analysis of what such skills involve. For teachers, educators and practitioners in the field of mathematics there is little doubt about the use they make and the value they place on estimation ..... What is not so obvious is:

(i) The method by which they estimate,
(ii) the variety of estimation methods which they as individuals use,
(iii) the time and the way in which they learned to estimate,
(iv) the mathematical ability of those who choose to use estimation.

"My hunch is that most of us come to estimation at a relatively late stage and at a time when we have gained some confidence in our ability to do the computation by the precise method".

My response to this last comment is that the introduction of the calculator has given us an immediate ability to "do the computation by the precise method". It is therefore high time that we introduced methods of estimation into the curriculum at the same time as we introduce the calculator; and the earlier we can do both these things successfully the better it will be.

The main concern of this paper is therefore to consider alternative methods of estimation so that the best and most appropriate methods may be taught. The

findings here put forward are intended to stimulate others to contribute their own views on the most appropriate methods of estimation to adopt. Any decision will of course have to be based on practical experiment, but it is to be hoped that the alternatives set out in this paper, together with the brief experiment so far undertaken by the author (4), will help to stimulate the great volume of experimentation that will be needed if the main thesis of this article is to be taken seriously.

2. Methods of Estimation

2.1 Addition

One of the most important concepts needed for estimation is "range". This seems to receive little attention in most formal mathematical syllabuses, so its development in any course on estimation must not be hurried. Fortunately the most obvious method of estimating the total of a column of numbers leads immediately to this concept.

$$\begin{array}{r} 2075 \\ 1489 \\ 693 \\ 2478 \\ \hline 5000 - 9000 \end{array}$$

In the example above it is obvious that the least the total could be is 5000: 2 + 1 + 2 = 5. The greatest it can be is then found by adding 1000 for each of the numbers in the set: this assumes that one "does not look at" the hundreds, tens and units figures so that each number could involve an extra 999.99 .... This gives the greatest the total could be as 9000, and the possible range is 5000 - 9000.

The accuracy of the estimate can be improved by adding the hundreds: 20 + 14 + 6 + 24 = 64. It should not be beyond most students' powers to do this mentally: it leads to a possible range of 6400 - 6800. Any discussion of why this is a 'better' estimate than 5000 - 9000 will begin to develop the students concept of a range and the fact that the "narrower" it is the better is the estimate.

The only ambiguity in this method lies in the units being used, in this case thousands or hundreds.

A practical use of this method of estimating lies in finding an error in a table

which has been added "across and down". This will have been done with a calculator, presumably, using the memory to add each set of totals to give the grand total. If two different grand totals are found it is usually possible to determine which digit is in error. If the error is a large one each total can be checked by finding its possible range and the error can, with luck, be identified. If however it is only a small error in, say, the thousands column it may be necessary to find exactly how much there is to carry from the hundreds. To do this one can of course add the hundreds figures, but it will also be necessary to estimate how much must be carried to the hundreds from the addition of the tens, otherwise we may not know exactly how many we have to carry from the hundreds to the thousands . This can be done by abler students if they can be shown that the "most likely" figure is a 5 so that the "most likely" amount to be carried from one digit column to the next is half the number of numbers in the set. The method is illustrated below:

```
  3251
  2484
  3376
  8249
  7528
  5309
  2837
 -----
 32134
```

The thousands figures add up to 30: the total shown is in the possible range 30000 - 37000 but we shall suppose we suspect an error in the 32. The hundreds figures add up to 27 but how much will there be to carry to the thousands? Here we "guess" that there will be half of 7 i.e. 3 or 4 to carry from the 10's figures, increasing the 27 to 30 or 31. The 1 in this total then makes it clear that there is a 3 to carry into the thousands column, showing that the 32 should have been 33.

Clearly a process such as this is far beyond the level of basic numeracy but it does indicate the sort of useful development that is possible in this area when more able students are involved.

A simpler test is of course available for the last figure in such an example. This is well worth teaching at the lowest levels because it will usually find the sort of error likely to have been made with a calculator.

This test simply consists of adding the last figures "modulo 10" i.e. in the

example above saying 1 and 4 is 5 and 6 is 1 and 9 is 0 and 8 is 8 and 9 is 7 and 7 is 4, thus checking that the final 4 in the total is correct.

2.2 Subtraction

$$\begin{array}{r} 8279 \\ -\ \underline{3654} \end{array}$$

The easiest estimate for a subtraction such as this is 5000, obtained by "ignoring" all the figures after the first. The possible range is 4000 - 6000 making the estimate of 5000 also the "most probable".

A better estimate can be obtained by subtracting 36 from 82, giving 4600 with a possible range of 4500 - 4700. When should one expect to get the estimate 5000 and when can one hope for 4600? The rule here is that if the answer can be "read out" mentally it is acceptable, but if it has to be written down <u>before</u> being read out then it is not acceptable.

This criterion emphasizes the need to learn how to determine the tens figure before the units figure when subtracting two figure numbers. This can be done by observing whether "borrowing" (or "re-grouping") is going to be necessary and if so subtracting 1 from the difference between the tens figures. The units figure can then be found exactly. But how often is the process actually taught in schools?

Indeed one of the first objectives in any course on mental arithmetic should be to enable the sum or difference of any two 2-figure numbers to be "read out" without having to write down the units figure first. The techniques for doing this do need to be taught, to some people at any rate. It seems a pity that such skills have been left largely to luck or individual enthusiasm in the past.

2.3 Means

If the range is one of the foremost concepts needed for developing methods of estimation then the mean of two numbers is certainly another. Given a possible range, the "best guess" is normally its mean. If you need to determine whether a number is high or low in a range then you may also need to estimate its mean.

This determination of "high or low" may incidentally be one of the best tests for number sense. Asked for a "medium-high" number in the range 40 - 50 most

of us would say 46 or 47 immediately. Not so most of my students in Papua New Guinea. The most improbable answers will in fact be given. It would be interesting to see if this particular skill has been picked up by the innumerate minority in the North, or whether they too need to be taught how to find a mean and so determine what is "high", "medium" or "low" in a given range. It seems likely that the continual exposure to numbers in the North will have provided the innumerate with this skill if not with many others.

To find the mean of two numbers with a calculator one would always add them and divide by 2 but this is not generally the best way when estimating. Smaller numbers are involved if instead the numbers are subtracted, the result is divided by 2 and this is then added to the smaller number. This also takes advantage of a good estimate being more easily obtained in subtraction than in addition (see above).

The only complication arises when an odd number has to be halved. Converting from $3\frac{1}{2}$ thousands to 3500 is not easy for our weaker students. So one may seize this opportunity to introduce another vital concept, that of "rounding". Half of 7 may be "rounded down" to 3 or "rounded up" to 4 or it may be given as a range 3 - 4 or it may be simply expressed as "3 or 4", where this is sufficient. Thus the estimated mean for 9582 and 2817 may be obtained by subtracting 2 from 9 to give 7, halving this to give 3 or 4 and then adding 3 or 4 thousands on to the 2817. This gives 5817 or 6817. By a happy coincidence this proves to be also the "possible range" for the mean, since the possible range for the difference is 6000 - 8000 so the possible range for half the difference is precisely 3000 - 4000. The mean cannot therefore be less than 5817 or more than 6817.

2.4 Multiplication

It is when we come to multiplication and division that the serious problems arise over the many alternative methods of estimating.

The most usual method is to take each number correct to 1 significant figure. Thus 1499 x 1499 would be treated as 1000 x 1000 giving an estimate of 1,000,000 whereas the true value is 2,247,001. The "error" or difference between the estimate and the true value is 1,247,001. That is 55% of the true answer or 124% of the estimate.

In order to compare different methods of estimating, a program was written by my son John M. Edwards for use on his Sinclair ZX Spectrum micro-computer.

Pairs of 4-figure numbers were chosen at random and their products estimated by using the "correct to 1 significant figure" method. The "error" in the estimate was expressed as a percentage of the true answer although in practice it could be argued that the error should really be assessed as a percentage of the estimate since that, not the true value, is all that one knows when one is using it to test the accuracy of an answer obtained in some other way.

As will be seen from the example above the error may be a much larger percentage of the estimated figure than it is of the true value but this is exceptional and it seemed more natural to express the errors as percentages of the true answer.

The form of printout on this particular computer is somewhat limited but its availability and the possibility of testing other methods of estimating at short notice far outweighed any advantages that would have accrued from using a main frame computer. The result shown below is quite intelligible and comparisons with other methods will be readily made later in this paper.

```
PROGRAM 1 RANDOMISE 7

  MEAN  -0.90200001
  Standard Deviation 12.747329

    0  30 94 16423 5  0  Group
  0  5  31 17388 17 2  0 Frequ.
   2  23 73 19459 15 2

500-                    No.of trials
                              1000

                 ----.....|||||||.....----
    -40-25-10 5  20 35 50 Midpts
  -50-35-20-5  10 25 40 55ranges
   -45-30-15 0  15 30 45
    Most positive %
44   2687 1550 4164850   6000000
43   1660 2530 4[illegible]2600   6000000
40   1528 4679 7149512   10000000
38   1505 2894 4355470   6000000
37   1542 5675 8750850   12000000
35   1806 2962 4460772   6000000
35   2686 1653 4443264   6000000
34   9886 1508 14908088  20000000
33   1519 3958 6012202   8000000
32   6554 1619 10610926  14000000
    Most negative %
-46  1483 1246 1847818   1000000
-45  1219 1493 1819967   1000000
-37  7486 1484 11109224  7000000
-37  1349 1186 1599914   1000000
-35  8190 1499 12276810  8000000
-34  1472 4100 6035200   4000000
-34  9485 1445 13705825  9000000
-32  1442 8213 11843146  8000000
-32  8152 1448 11804096  8000000
-32  1441 8122 11703802  8000000
```

Program 1.
Method of estimation being tested

Each number is replaced by the nearest 1 significant figure number and these two numbers are multiplied together. e.g. 6153 x 3576. 6000 x 4000 = 24,000,000

Result : 95% of the 4000 estimates tested had errors between - 28% and + 26%

The mean error in the estimates for this random set of 1000 number-pairs is - 0.9 confirming that there is no serious bias in the method of estimating that is being used. The standard deviation of 12.7 will be the chief parameter used in comparing this with other methods. If the distribution were normal we would expect 95% of the % errors in the estimates to lie between - 26% and + 24% (mean ± 1.96 s.d.).

The group frequency figures given in the printout for this and other runs of the same program can however be used to give more precise figures for this range. In this case a total of 4000 number-pairs were trialled and in 95% of these cases the estimates had percentage errors of between - 28% and + 26%. The difference between these limits and those found above would be due to the particular set of "random numbers" used in the run for which the output is shown above. (Each "set" of random numbers is defined on the Spectrum by one number, in the case of this output, 7.)

To read these frequencies from the output, it should be noted that the mid-points of the groups are listed beneath the bar chart and the frequency for each group is listed above that. The first figure of the frequency, and the bar of the chart are placed directly above the first figure of the group's mid-point. Thus the number of estimates with errors between - 12.5 and - 7.5% is found by identifying the 1 of the "-10" which is the mid-point of the range, looking above it for the corresponding bar of the bar chart, and above that for the first figure 9 of the frequency 94.

If one is content with the fact that 95% of one's estimates are likely to be less than 29% off the true value then this method of approximating can be considered adequate.

If an answer obtained by calculator has the decimal point in the wrong place then it will differ from the true answer by at least 90% and possibly 900% of the true answer, which is clearly much more than the possible error in the estimate. It does not however follow automatically that finding an estimate in this way will necessarily identify an error in the answer being tested.

Suppose we take the "worst case" mentioned above, 1499 x 1499. The estimate for this will be 1,000,000. The correct answer would be 2,247,001 and the answer obtained by misplacing the decimal point might be 224,700.1. This differs from

the estimate by 78% of the estimate. But the correct answer differs from the estimate by 1,247,001, that is by 125% of the estimate. Thus, not knowing the true answer, one could not say with certainty that because the answer obtained differed from the estimate by 78% it was necessarily wrong.

Admittedly this is an exceptional case, but I am not myself convinced that errors in the use of the calculator are in fact often found by this method. Most calculator users have a whole list of estimating procedures available to them which they would use when appropriate. Although in fact most people will quote "finding errors in calculator results" as being a prime reason for learning to estimate, I suspect that more errors are found by "practical" than by "arithmetical" estimating. For the less numerate calculator user it is generally easier to carry out the calculation twice or to use an in-built checking procedure, than it is to estimate every answer and re-work it if it differs "substantially" (whatever that may mean) from the estimate.

As I have explained earlier however, I believe there are much more important reasons for learning to estimate than just being able to find errors in using the calculator. The prime reason for learning to estimate is, to my mind, the development of "number sense". But there are many situations in real life where even if one did have a calculator in one's pocket, one would prefer to obtain a rough answer by estimation. These are the real reasons for developing estimation skills and these reasons justify a better method of estimating than the one analysed above in Program 1.

An obvious criterion for selecting an estimation procedure is that of "simplicity". But there are two sorts of simplicity. Simplicity of concept and simplicity of operation. If we are hoping that a student will understand what is happening then one needs a method which is simple in concept, but if we are primarily concerned with its use then simplicity of operation is paramount. In the case of my Papua New Guinean students the whole concept of estimation is strange to them: even teaching how to give a number correct to 1 significant figure takes a long time and for some it has eventually to be reduced to rote learning "if the second figure is 5, 6, 7, 8 or 9 ....."

I have concluded therefore that two different types of course are required. A first course in which the concepts are as simple as possible even if the operation is slow and requires some writing down of intermediate steps. Such a course was

produced and trialled on a class of 24 government officers in March 1983 (4). The course itself consisted of 20 units. These were pre-recorded on 10 audio-cassettes which could be then used either for self-teaching or as reinforcement of class-teaching. Memo-sheets and tests were also prepared.

This course, which was designed primarily to give an understanding of the concepts involved in estimation will be followed by a course "Understanding Statistics" (5) which will develop improved methods of estimation based on these concepts. The "Calculate and Estimate" course aims at simplicity in concept, "Understanding Statistics" will give priority to simplicity of operation.

To return to alternative methods of estimating products: In the "Calculate and Estimate" course (4) we began by teaching the concept of the "1 significant figure range" e.g. for 3261 it is 3000 - 4000. Having found the 1 significant figure range for each number we then multiplied the lower ends of the ranges together to obtain the least the product could be and we multiplied the upper ends of the ranges to find the most the product could be. This gave the Possible Range for the product.

We then went further and determined whether each number was Low, Medium or High in its own 1 significant figure range, and so decided whether we wanted a low-low, medium or medium-high etc. result within the possible range.

The decision whether a given number was "low", "medium" or "high" within its 1 significant figure range could be made on the basis that if the second figure of the number was 0, 1 or 2 then it was "low", if the second figure was 7, 8 or 9 it was "high" and otherwise it was "medium". Determining what was low-low or medium-high etc. in a given "possible range" was much more difficult. It involved finding the mean and sometimes even the mean of the upper half of the range in order to explain why an answer was or was not acceptable.

As an example: 265 x 49 lies between 200 x 40 and 300 x 50, so its "possible range" is 8000 - 15000; 265 is "medium" and 49 is "high" so we want a "medium-high" number in this range; the mean of 8000 and 15,000 is 11,000 or 12,000 so 13,000 would be a good answer, while 14,000 would be a "high-high" answer.

As an exercise in developing concepts and as a means of finding a possible range the method was acceptable but in the proposed "Understanding Statistics" course

it will be important to provide a method which is simpler to operate.

A possible improved method is analysed in the Program 2 print-out below. This method consists in defining numbers whose second figures are 7, 8, 9, 0, 1 or 2 as "close-to". If the first number is "close-to" *it is replaced* by the nearest 1 significant figure number. Then the estimate is obtained by multiplying the first figure of the second number by it and adding on whatever would be carried by multiplying the second figure by it. So 524 x 738 would be worked thus: 524 is "close-to" 500. 5 x 7 is 35 and 5 x 3 is 15, 1 to carry, 35 and 1 is 36, estimate 360 000 (one zero for each figure after the first ones).

If the first number was not "close-to" but the second number was, then the latter was replaced by a 1-significant figure number and the same process applied.

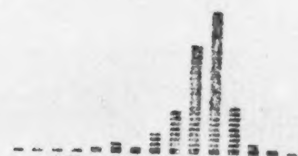
If neither number was "close-to" then the possible range was found as for Program 1 and its mean was taken as the estimate, e.g. for 1388 x 3351 the possible range is 3,000,000 - 8 000,000 with a mean of 5,000,000 or 6 000,000. For simplicity the computer program used 5 500.000 although the student would not necessarily be expected to do this.

```
PROGRAM  2 RANDOMISE 7

    MEAN   -3.657
    Standard Deviation 7.9751709

       2  21 1051211  0  0  Group
    3  3  13 28121 0  0  0  Frequ.
     2  7  46 3704  0  0

500-                  No.of trials
                         1000
```



```
   -40-25-10 5  20 35 50 Midpts
  -50-35-20-5  10 25 40 55ranges
   -45-30-15 0  15 30 45
    Most positive %
20   1310  7335  9608850   11500000
15   1715  8641  14819315  17000000
15   9451  1326  12571806  14500000
15   1714  8606  14750684  17000000
13   1352  9510  12857520  14500000
12   1773  3012  5340276   5000000
12   2381  2431  5788211   6500000
11   2301  9364  21546564  24000000
11   2340  3469  8117460   9000000
11   1667  1355  2258785   2500000
    Most negative %
-52  1282  1616  2071712   1000000
-49  1235  1600  1976000   1000000
-49  1225  1616  1979600   1000000
-46  1246  1483  1847818   1000000
-45  1219  1493  1819967   1000000
-42  1065  1614  1718910   1000000
-42  1237  2810  3475970   2000000
-37  1186  1349  1599914   1000000
-36  1104  2819  3112176   2000000
-36  1279  5980  7648420   5000000
```

Program 2.

Method of Estimation being tested

If the second figure of the first number is 7,8,9,0,1 or 2 the number is called "close-to" and is replaced by the nearest 1 significant figure number. The first figures of the estimate are obtained by multiplying this by the first figure of the second number and adding to this whatever would have been carried if the second number had been multiplied also. e.g. 4762 x 5384. 4 x 5 = 20 and 4 x 3 = 12 so there is 1 to carry. Estimate 21,000,000. If the first number is not close to but the second is, then the above process is reversed e.g. 3564 x 6271. 6 x 3 = 18 and 6 x 5 = 30, so there is 3 to carry. Estimate 21 000 000. If neither number is close-to then the possible range is found as in Program 1 and the mean is taken. e.g. 2674 x 8417. 2 x 8 = 16, 3 x 9 = 27, mean = 21 or 22. Estimate 21 000 000 or 22 000 000 (21,500,000 on the computer).

Result : 95% of the 4000 estimates tested had errors between - 24% and + 9%

With a mean % error of - 3.7 there is a clear bias towards under-estimating but a standard deviation of 8 shows a marked improvement over Program 1. 95% of the 3500 estimates tested had errors of between - 24% and + 9%.

What will be noticed about this method is that it leads on logically from the method described previously: "Low" and "High" numbers having second figures 0, 1, 2 or 7, 8, 9 became "close-to" while the estimate for the product of two "medium" numbers is taken as the mean.

The disadvantage of the method of Program 2 lies in the fact that one cannot expect a mean to be found without first writing down the numbers of which the mean is required. Thus the method is not strictly within our definition of an acceptable method of estimating, i.e. one involving only mental work.

```
PROGRAM  3 RANDOMISE 7

   MEAN  -3.893
   Standard Deviation 7.9242382

    3  21 1101120  0  0  Group
   3  3  13 28519 0  0  0 Freq.
    2  7  46 3703  0  0

500-                   No.of trials
                           1000
```

```
    -40-25-10 5  20 35 50 Midpts
  -50-35-20-5  10 25 40 55ranges
   -45-30-15 0  15 30 45
    Most  positive %
15   1715 8641 14819315 17000000
16   1714 8606 14750684 17000000
14   1310 7335 9608850  11000000
12   1773 3012 5340276  6000000
11   2301 9364 21546564 24000000
11   9481 1326 12571806 14000000
11   2340 3469 8117460  9000000
11   1772 4556 8073232  9000000
10   2324 9364 21761936 24000000
10   1746 9860 17215560 19000000
    Most  negative %
-52  1282 1616 2071712  1000000
-49  1235 1600 1976000  1000000
-49  1225 1616 1979600  1000000
-46  1246 1483 1847818  1000000
-45  1219 1493 1819967  1000000
-42  1065 1614 1718910  1000000
-42  1237 2810 3475970  2000000
-38  1234 2626 3240484  2000000
-37  1186 1349 1599914  1000000
-36  1104 2819 3112176  2000000
```

<u>Program 3</u>

<u>Method of Estimation being tested</u>

If one number is "close-to" the method of Program 2 is used. If not, the first figures of the estimate are obtained by multiplying the first figures of the two numbers and adding their mean (rounded up). e g 2479 x 9627. 2 x 9 = 18. Mean of 2 and 9 (rounded up) is 6. Estimate 24,000,000.

<u>Result</u> 95% of the 4000 estimates tested had errors between - 24.5% and + 8%

The method of Program 3 was developed to try to overcome this objection. It was noted that $(a + \frac{1}{2})(b + \frac{1}{2}) = ab + \frac{1}{2}(a + b) + \frac{1}{4}$, i.e. just over the sum of the product and the mean. If a and b represent the first figures of the two numbers being multiplied then one should obtain a reasonable estimate for the "neither close-to" case by multiplying these numbers and adding their mean. As they are single digits their mean can be found mentally and to reduce the error due to omitting the $\frac{1}{4}$ one can round up the mean whenever the numbers differ by an odd number. Thus for 1388 x 3351 this method gives:- 1 x 3 + (mean of 1 and 3) x 1 000 000 which is 5 000 000.

The analysis is given in the printout above. It is pleasingly similar to that for Program 2, the means and standard deviations differing only slightly and the "worst" examples being the same.

These "worst examples" are shown in the lists headed "Most positive %" and "Most negative %". In each case the first number is the error %, the next two are the numbers being tested, the next is the true value and the last is the estimate.

It will be noted that the worst errors occur with numbers starting 12, which are replaced by 1000. This is then multiplied by a second number e.g. 1616 but the method for this is that described in Program 2 which works out as 1 000 000, since 1 x 6 is 6 so there is nothing to carry and 1 x 1 is 1 giving the estimate 1 000 000. Of course there is no reason why one should not multiply exactly by 1000, which would give a better estimate of 1616 000. Similarly when a number is "close-to" 10 000 the estimate can be better found by multiplying by this exactly.

Program 13 shows the effect of this modification.

This shows a considerable improvement over Programs 2 and 3, with a mean error of - 3.2% and a standard deviation of only 6.4. 95% of the 5000 number pairs tested with this program have estimates with errors of between - 16% and + 7.5%

This last method would therefore seem to be the most suitable method of estimating a product for anyone who knows their multiplication tables and is prepared to learn some simple techniques. It is offered in this paper in the hope that some even better method may be suggested that still meets the criteria proposed. It may not be at all simple to understand but it is quite simple enough to operate once the user is familiar with the means of pairs of single figure numbers.

```
PROGRAM  13 RANDOMISE 7

   MEAN  -3.247
   Standard Deviation 6.3732246

     0  11 1121170  0  0  Group
   0  0  22 29119 0  0  0 Frequ.
     0  1  53 3713  0  0

500-                          No.of trials
                                  1000

                  ---------.---------
                  -40-25-10 5  20 35 50 Midpts
                -50-35-20-5  10 25 40 55Ranges
                 -45-30-15 0  15 30 45
                 Most positive %
15   1715 8641 14819315 17000000
15   1714 8606 14750334 17000000
14   1323 7395 9808850  11000000
12   1773 3012 5340276  6000000
11   2301 9364 21546564 24000000
11   9481 1326 12571806 14000000
11   2340 3469 8117460  9000000
11   1772 4556 8073230  9000000
10   2324 9364 21761936 24000000
10   1746 9860 17215560 19000000
     Most negative %
-28  3020 1373 4158317  3000000
-27  2150 1278 2747700  2000000
-27  2150 1267 2724050  2000000
-26  3050 1337 4077850  3000000
-24  2025 1952 3952800  3000000
-24  2017 1299 2620083  2000000
-24  2850 1383 3955300  3000000
-24  2271 3496 7939416  6000000
-24  1960 1350 2646000  2000000
-23  1297 7353 9536841  7353000
```

Program 13

Method of Estimation being tested

As for Program 3 but when the first number is "close-to" 1000 or 10,000 the other number is multiplied by this exactly. e.g. 1295 x 4638. Estimate 4 638 000.

If the first figure is not "close-to" but the second is close to 1000 or 10 000 the first number is multiplied exactly by it. e.g. 3473 x 9815. Estimate is 34 730 000

Result : 95% of the 5000 estimates tested had errors between - 16% and + 7.5%

2.5 Division

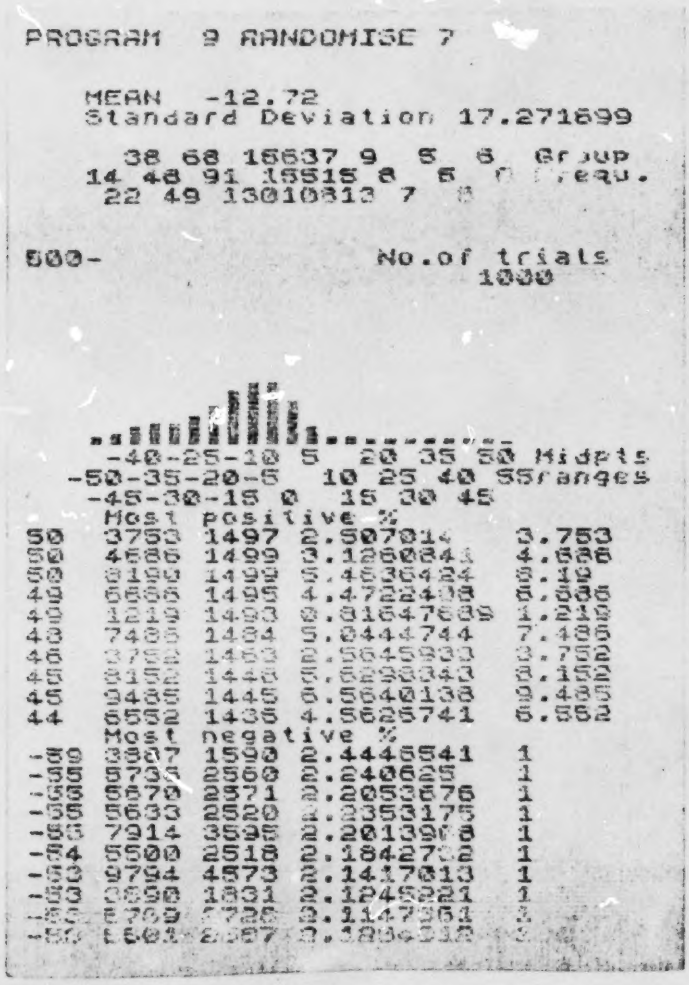
It is in this process of division that the most serious problems of estimation arise. The ability of the students to divide mentally is the principal restriction. "Knowing the multiplication tables" is assumed but obtaining a second figure in the quotient implies being able to multiply, subtract, carry and divide, all in the head. Even deciding whether the first figure is nearer the integer above or the integer below would imply subtracting to find the remainder, and then determining whether this was more or less than half the divisor.

Ideally one would wish to avoid both these processes in the case of numerically weak learners. Program 9 shows the best that it seems possible to do with these restrictions.

The divisor is here rounded to the nearest 1 significant figure number and the dividend is divided by it, to 1 significant figure and rounded down. In other

words for 5602 ÷ 2579 one says 2579 is nearest to 3000 and 5602 ÷ 3000 gives a first figure of 1. So the estimate is 1 assuming as always that the position of the decimal point in the answer can be correctly determined.

The exception to this is that where the division is by 1 (or a power of 10) we take all four figures of the dividend.



```
PROGRAM  9 RANDOMISE 7

   MEAN  -12.72
   Standard Deviation 17.271699

     38 68 15637 9   5   6   Group
   14 48 91 15515 8   5   0 Frequ.
    22 49 13010813 7   8

500-                    No.of trials
                              1000

   ..:::::::::::::::::::::::::::::::::::..
    -40-25-10 5   20 35 50 Midpts
  -50-35-20-5   10 25 40 55ranges
   -45-30-15 0   15 30 45
    Most positive %
50   3753  1497  2.507014     3.753
50   4686  1499  3.1260841    4.686
50   8190  1499  5.4636424    8.19
49   6686  1495  4.4722408    6.686
49   1219  1493  0.81647685   1.219
48   7486  1484  5.0444744    7.486
46   3752  1463  2.5645933    3.752
45   8152  1448  5.6298343    8.152
45   9485  1445  6.5640138    9.485
44   6552  1435  4.5625741    6.552
     Most negative %
-59  3807  1590  2.4446541    1
-55  5735  2560  2.240625     1
-55  5670  2571  2.2053676    1
-55  5633  2520  2.2353175    1
-53  7914  3595  2.2013918    1
-54  5500  2518  2.1842732    1
-53  9794  4573  2.1417013    1
-53  3690  1831  2.1245221    1
-52  2709  1726  2.1147261    1
-50  1501  2287  2.1034318    1
```

Program 9

Method of Estimation being tested

The divisor is rounded to the nearest 1 significant figure number and the estimate is found by dividing as far as 1 significant figure, rounded down. e.g. 9685 ÷ 1847. 1847 correct to 1 significant figure is 2000. 2 into 9 goes 4. Estimate 4. Exception: when the divisor is rounded to 1000 or 10,000 the answer is given exactly. e.g. 5763 ÷ 1428. Estimate 5.763.

Result : 95% of the 2500 estimates tested had errors between - 46% and + 27%

The result is a mean of - 13% and a standard deviation of 17. The 2500 number pairs tested had estimates with errors of between - 46% and + 27%.

One immediately sees however where the worst errors lie. The "most negative %" list consists entirely of estimates consisting of 1. Now this happens to be the one case where a second figure is reasonably easy to find. One has merely to subtract the divisor, "carry" to form the next dividend and see how many times the divisor goes into it. In the above example, 5602 ÷ 3000, we have only to subtract the 3 from the 5, form the dividend 26 and decide that 3 goes into this

8 times, giving an estimate of 1·8: a great improvement on 1.

This improved method is tested in Program 10.

```
PROGRAM  10 RANDOMISE 7

   MEAN  -5.738
   Standard Deviation 12.971559

     4  38 18074 12 5  6  Group
   0  14 62 22527 8  5  0 Frequ.
    0  17 12417115 7  6

500-                  No.of trials
                          1000

                 ----------------------
     -40-25-10 5  20 35 50 Midpts
   -50-35-20-5  10 25 40 55ranges
    -45-30-15 0  15 30 45
     Most positive %
50   3753 1497 2.507014    3.753
50   4686 1499 3.1260841   4.686
50   8190 1499 5.4636424   8.19
49   6686 1495 4.4722408   6.686
49   1219 1493 0.81647689  1.219
48   7486 1484 5.0444744   7.486
46   3752 1463 2.5645933   3.752
45   8152 1448 5.6298343   8.152
45   9485 1445 6.5640138   9.485
44   6552 1436 4.5626741   6.552
     Most negative %
-39  9886 1508 6.5557029   4
-38  5756 1778 3.2373453   2
-38  1472 4560 0.32280702  0.2
-38  5891 1838 3.2051143   2
-37  6340 2609 3.1966271   2
-36  5891 1883 3.1285183   2
-36  7220 1548 4.6640827   3
-35  2330 7554 0.30844586  0.2
-35  8302 2697 3.0782351   2
-35  6972 2936 3.0558583   2
```

Program 10.

Method of Estimation being tested

As for Program 9 but when the first figure of the quotient is 1, a second figure is found, e.g. 6389 ÷ 4267. 4 into 6 goes 1 with remainder 2. 4 into 23 goes 5. Estimate 1.5.

Result 95% of the 2500 estimates tested had errors between - 31% and + 27%.

It results in a substantial reduction in the underestimates, with the mean up to - 5·7 and the standard deviation down to 13. 95% of the 2500 number pairs tested had estimates with errors between - 31% and + 27%.

We see that the worst errors now arise from treating divisors like 1497 as if they were 1000, but most of the ten worst under-estimates now arise in cases where the first figure of the quotient is 2.

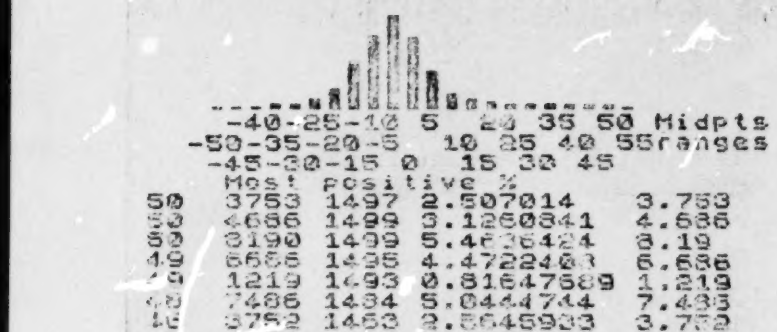
One is therefore tempted to ask the learners to go further and obtain two significant figures in the quotient whenever the first figure is a 1 or a 2. This is slightly more difficult, but it is one step towards doing it in every case. The result is shown in Program 11.

PROGRAM 11 RANDOMISE 7

MEAN -4.105
Standard Deviation 11.995748

1 21 18792 13 5 6 Group
0 4 46 24231 8 5 0 Frequ.
0 7 11918218 7 6

500- No.of trials 1000



	Most positive %			
50	3753	1497	2.507014	3.753
50	4686	1499	3.1260841	4.686
50	8190	1499	5.4636424	8.19
49	6686	1495	4.4722403	6.686
49	1219	1493	0.81647589	1.219
48	7486	1484	5.0444744	7.486
46	3752	1463	2.5645933	3.752
46	8152	1448	5.6298343	8.152
45	9485	1445	6.5640138	9.485
44	6552	1435	4.5636741	6.552
	Most negative %			
-39	9386	1503	6.5557029	4
-36	7229	1548	4.6640827	3
-35	7389	1719	4.5892961	3
-33	9087	1530	5.9382157	4
-33	9540	1593	5.9887005	4
-32	7221	1629	4.4327808	3
-29	9889	1748	5.6573227	4
-29	9305	1572	5.9160957	4
-29	7222	1720	4.1988372	3
-29	1940	4588	0.42463352	0.3

Program 11

Method of Estimation being tested

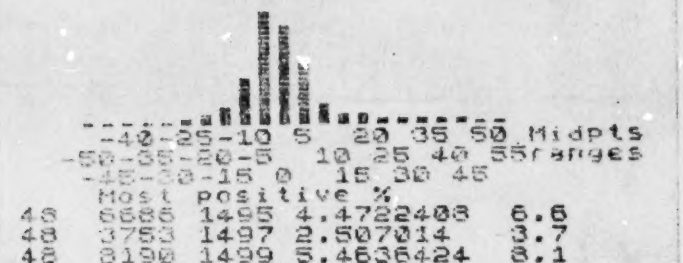
As for Program 10 but when the first figure of the quotient is 2 a second figure is also found e.g. 7924 ÷ 2614. 3 into 7 goes 2 with remainder 1. 3 into 19 goes 6. Estimate 2.6

Result : 95% of the 2500 estimates tested had errors between - 24% and + 27%.

The mean has been raised to - 4·1 and the standard deviation reduced to 12. 95% of the 2500 number pairs tested had estimates with errors between - 24% and + 27%. Predictably most of the 10 worst underestimates have been replaced by underestimates starting with 3 or 4.

Going to the limit in this direction. Program 4 shows the result of taking all quotients to 2 significant figures, rounded down of course.

This produces a mean of - 0·5 and standard deviation of 10. 95% of the 3500 number pairs tested had estimates whose errors were between - 18% and + 27%.



```
PROGRAM  4 RANDOMISE 7

   MEAN  -0.47600001
   Standard Deviation 10.215254

     0  6  11515621 6  3  Group
   0  0  17 20748 9  6  0 Frequ.
    0  0  41 25519 5  6

500-                    No.of trials
                              1000

                 ----------
     -40-25-10 5  20 35 50 Midpts
   -50-35-20-5  10 25 40 55ranges
    -45-30-15 0  15 30 45
     Most positive %
48   6686 1495 4.4722408   6.6
48   3753 1497 2.507014    3.7
48   8190 1499 5.4636424   8.1
47   4686 1499 3.1260841   4.6
47   7486 1484 5.0444744   7.4
47   1219 1493 0.81647689  1.2
44   8152 1448 5.6298343   8.1
44   3752 1463 2.5645933   3.7
43   9486 1445 6.5640138   9.4
42   5520 1426 3.8709677   5.5
     Most negative %
 27  4153 1528 2.7214941   2
-25  2657 1550 1.7335484   1.3
-25  9886 1508 6.6557029   4.9
-24  8153 1551 5.2568086   4
-24  9087 1530 5.9392157   4.5
-23  7220 1548 4.6640827   3.6
-22  4402 1560 2.8217949   2.2
-22  4287 1596 2.6860902   2.1
-22  3887 1590 2.4446541   1.9
-22  9540 1593 5.9387006   4.7
```

Program 4

Method of Estimation being tested

The divisor is rounded to the nearest 1 significant figure number and the division carried as far as 2 significant figures, rounded down, in all cases. e.g. 4732 ÷ 1564. 47 ÷ 2 gives 23. Estimate 2.3

Result 95% of the 3500 estimates tested had errors between - 18% and + 27%.

These then are our options with this basic method of estimating by taking the divisor to its nearest 1 significant figure number: If the quotient is given to 1 significant figure (rounded down) except when the divisor is 1 (or a power of 10), 95% of the estimates have errors of between - 46% and + 27%. (Program 9)

If the quotient is given to 2 significant figures when its first figure is 1 then 95% of the estimates have errors of between - 31% and + 27%. (Program 10)

If the quotient is given to 2 significant figures when its first figure is 1 or 2 then 95% of the estimates have errors between - 24% and + 27%. (Program 11).

If the quotient is always given to 2 significant figures (rounded down) then 95% of the estimates have errors of between - 18% and + 27%. (Program 4)

All these however leave a major source of error uncorrected. In taking our divisor correct to the nearest 1 significant figure number we will expect large errors when it lies midway between two such numbers. This of course happens particularly when it lies midway between 1000 and 2000, since the proportionate difference is here greatest. As Program 4 shows, the worst errors all arise from divisors between 1400 and 1600.

We are certainly not going to expect our students to learn their 14 or 15 times tables, so there seems little alternative here but to think in terms of finding a mean between dividing by 10 and dividing by 20.

This in fact is what was done in the course run at the Administrative College in March 1983, referred to above (4).

We used a similar method to that used for multiplication, but with necessary differences. Firstly, it was not necessary to round the dividend in any way: it is no easier to divide into 5000 than it is to divide into 5349. So only the divisor was rounded to the 1 significant figure numbers above and below.

Here however came the first surprise for the students: dividing by the larger number produced a smaller result. So to get the numbers of the possible range in the right order one had to start by dividing by the 1 significant figure number <u>above</u> the divisor: this, rounded <u>down</u> to 1 significant figure, gives the <u>least</u> the answer can be.

That was relatively easy, but to get the <u>most</u> the answer could be one had not only to divide by the 1 significant figure number <u>below</u> the divisor, but one also had to round it <u>up</u>. As example will illustrate this: for 8159 ÷ 1652 we first do 8159 ÷ 2000 and round down, giving 4, then we do 8159 ÷ 1000 and round up giving 9. The possible range was thus 4 - 9. In practice it was found that the final rounding up was one of the most difficult parts of the process to teach.

Where the first figure of the quotient was a 1 a second figure was found, as in Program 10. For example to find 4172 ÷ 2968 we would first do 4172 ÷ 3000 and round down giving 1·3, then 4172 ÷ 2000 and round up giving 3. The possible range is 1·3 to 3.

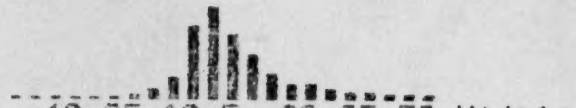
When multiplying we had estimated whether the result would be low, medium or high etc. within the possible range but this would be impracticable with division because a "high" divisor would result in a "low" quotient. In all cases therefore we just took the mean of the possible range. In practice we would say the mean of 4 and 9 was "6 or 7" but in Program 5 where the results of this method are analysed the true mean (e.g. 6·5) was in fact used so the results will be rather more accurate than could have been expected in practice.

PROGRAM 5 RANDOMISE 7

MEAN 4.8469999
Standard Deviation 12.8586

	0	2	56	17133	14	6	Group
0	0	7	19111432		11	6	Frequ.
	0	0	21	24762	22	5	

500- No.of trials 1000



-40-25-10 5 20 35 50 Midpts
-50-35-20-5 10 25 40 55ranges
-45-30-15 0 15 30 45

	Most positive %			
93	2025	1952	1.0173975	2
73	3092	1945	1.5897172	2.75
68	3226	1935	1.6666667	2.8
62	5026	1954	3.0339304	3
56	4155	1858	2.2378902	3.5
53	6156	1867	3.263381	5
52	4425	1919	2.3058385	3.5
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	most negative %			
-28	7180	1003	7.4576271	3.5
-23	8400	1000	8.4	3.5
-21	5382	1162	5.0619621	4
-21	8668	1050	8.2552381	6.5
-21	8547	1042	8.2024952	6.5
-18	1614	1065	1.5154493	1.25
-18	7493	2044	3.6658513	3
-18	8013	1011	7.928516	6.5
-18	6280	1031	6.0911735	5
-17	3993	2000	1.9995	1.65

Program 5 :

Method of Estimation being tested

The divisor is rounded up to the 1 significant figure above and the quotient to 1 significant figure (or 2 if the first figure is a 1) is rounded down. This gives the least it can be. The divisor is then rounded down to the 1 significant figure below and the quotient is found as before but rounded up. This gives the greatest the answer can be. The mean of the possible range is taken as the estimate. e.g. 7217 ÷ 3689. 7217 ÷ 4000 = 1.8. 7217 ÷ 3000 = 3. Estimate 2.4.

Result : 95% of the 2500 estimates tested had errors between - 13% and + 38%.

The results are about as good as were obtained on Program 10 where we took the divisor to its nearest 1 significant figure number and found the quotient to 2 figures when its first figure was a 1. 95% of the 2500 number-pairs tested had estimates with errors between - 13% and + 38%.

The main sources of error seem to be with divisors between 1800 and 2000. This sometimes follows from the rounding up of 2 to 3: since we go to 2 figures when the first is 1 there is no question of rounding up from 1 to 2 which would otherwise have greatly increased the errors.

There are other reasons for the errors being large and there is general overestimating shown by the mean error of nearly 5%, but the graph is skew with many more errors over 20% than under - 20%.

Further analysis is however not particularly profitable since the method was used primarily to develop concepts rather than to obtain quick and accurate results.

Program 6 shows the effect of using the "close-to" technique suggested above for multiplication. If the second figure of the divisor is a 7, 8, 9, 0, 1 or 2 then the divisor is replaced by the nearest 1 significant figure number and the quotient is given to 2 significant figures, rounded down. Where the divisor is <u>not</u> "close-to", the result is obtained as in Program 10 by taking the mean of the possible range, quotients being worked out to 1 significant figure, or to 2 significant figures if the first figure of the quotient is a 1.

```
PROGRAM  6 RANDOMISE 7

   MEAN  1.035
   Standard Deviation 7.6988884

     0   0   52 15224 1   0   Group
   0   0   0   26456 23  0   0 Frequ.
     0   0   11 38424 9   0

500-                    No.of trials
                              1000
                 ##
                ####
               ######
              ########
             ##########
            ############
           ##############
          ################
         ##################
        ####################
       ######################
      ########################
     ##########################
    ############################
   ##############################
   -40-25-10 5   20 35 50 Midpts
  -50-35-20-5   10 25 40 55ranges
  -45-30-15 0   15 30 45
   Most positive %
37   4168  2647 1.5746128   2.15
32   2668  1653 1.6261343   2.15
32   7089  1695 4.1803009   5.5
31   4289  1600 2.6666667   3.8
30   4287  1596 2.6860902   3.5
29   7350  1299 5.6531986   7.3
29   2017  1299 1.5527329   2
29   4168  1526 2.7214941   3.5
29   4421  1633 2.7072872   3.5
29   6046  3465 1.7448773   2.25
   Most negative %
-16  6135  1710 3.5877193   3
-16  1355  1701 0.79659024  0.87
-15  6956  1739 4           3.4
-15  4169  2716 1.5349779   1.3
-15  9956  1734 5.1649366   4.4
-15  7889  1719 4.5892961   3.9
-14  5755  1778 3.2373453   2.8
-14  8823  1732 5.0941109   4.4
-14  7222  1720 4.1986372   3.5
-14  4555  1772 2.5711081   2.2
```

Program 6.

Method of Estimation being tested

If the divisor is "close-to" (i.e. if its second figure is 7,8,9,0,1 or 2) it is rounded to the nearest 1 significant figure number and the quotient found by division to 2 significant figures (rounded down). e.g. 2136 ÷ 3974. Estimate 2136 ÷ 4000 = 0.53. If the divisor is not "close-to" the estimate is found as in Program 5.

<u>Result</u> : 95% of the 2500 estimates tested had errors between - 11% and + 24%.

This gives us the best result so far for division, a mean % error of 1 and standard deviation of 7·7. 95% of the 2500 estimates tested have errors between - 11% and + 24%.

The method would be good for an able student but it does depend on being able to divide to 2 significant figures in some case and in others it means finding the mean of a possible range. We therefore search for a method that might be easier to carry out mentally.

To find a method that avoids having to find a mean we consider the algebraic representation of a divisor that is midway between two integers.

We suppose that the divisor is between 1 and 10 and that its first digit is b: its 'most likely' value is then $b + \frac{1}{2}$. If the dividend is "a" the best estimate would be $\frac{a}{b + \frac{1}{2}}$. If however we divided simply by the first figure of the divisor, b, then our "first estimate" would be $\frac{a}{b}$. The difference between this "first estimate" and the "best estimate" is $\frac{a}{b} - \frac{a}{b + \frac{1}{2}} = \frac{a}{2b(b+\frac{1}{2})}$. This is approximately equal to the first estimate divided by 2b.

We then obtain the following rule: divide by the first figure of the divisor as far as 2 figures, then divide this 2-figure number by the first figure of the divisor again, halve the result and subtract it from the 2-figure number (all divisions being rounded down). Put the decimal point in the right place and you have the estimate.

This is not quite as complicated as it sounds. For example: given 4927 ÷ 3586 we divide 49 by 3 giving 16, then divide this by 3 again giving 5 and also halve it, giving 2. Subtracting this from 16 we get 14. The estimate is 1·4.

The analysis is given in Program 7.

```
PROGRAM  7 RANDOMISE 7

   MEAN   -2.385
   Standard Deviation 7.4912519

     0   13 76 96 11 0   0   Group
  0   4    11 29515 8   0   0 Frequ.
    0   12 19 4344   2   0

500-                          No.of trials
                                 1000

   ---------------------------
    -40-25-10 5   20 35 50 Midpts
  -50-35-20-5   10 25 40 55ranges
   -45-30-15 0   15 30 45
    Most positive %
29   7350  1299 5.6581986   7.3
29   2017  1299 1.5527329   2
27   1615  1282 1.2605304   1.6
27   9500  1272 7.4686535   9.5
27   7463  1289 5.8052754   7.4
25   2150  1276 1.6823161   2.1
24   1600  1235 1.2955466   1.6
24   9490  1249 7.5980785   9.4
24   2150  1267 1.6969219   2.1
23   7040  1241 5.6216372   7.3
     Most negative %
-35  7684  1318 5.8317906   3.9
-34  7335  1310 5.5992366   3.7
-33  9217  1338 6.8886398   4.6
-33  4151  1331 3.1187077   2.1
-32  2850  1385 2.0533141   1.4
-32  9510  1352 7.0340237   4.8
-32  9215  1372 6.7186589   4.6
-31  7751  1362 5.6908957   3.9
-31  8180  1385 5.9061372   4.1
-30  4800  1400 3.4285714   2.4
```

Program 7

Method of Estimation being tested

If the divisor is "close-to" it is rounded to the nearest 1 significant number and the quotient is found by dividing to 2 significant figures (rounded down), as in Program 6.
If the divisor is not close-to, the dividend is divided by the first figure of the divisor as far as 2 significant figures. This "first estimate" is then again divided by the first figure of the divisor and is also halved. The result is subtracted from the "first estimate". All divisions are rounded down. e.g.
4731 ÷ 5697. 473 ÷ 5 = 94.
94 ÷ 5 = 18. 18 ÷ 2 = 9.
94 − 9 = 85. Estimate 0.85.

Result 95% of the 3500 estimates tested had errors between - 24% and + 12%

With a mean of -2·4 and standard deviation of 7·5 this is similar to Program 6 which it was designed to replace, although it tends to under-estimate where Program 6 over-estimated. 95% of the 3500 estimates tested have errors between - 24% and + 12%.

This procedure is still very complicated however so we look for a simpler one. We continue to use the 2 significant figure quotient for the "close-to" numbers but for those which are not close-to we return to the finding of a mean. This time however we ease the problem a little by rounding down instead of up when finding the upper end of the range and try to compensate by rounding up instead of down when taking the mean. We also simplify it by only going to 1 significant figure in each case. So for 8967 ÷ 1645 we would find 8967 ÷ 2000 giving 4 and 8967 ÷ 1000 giving 8 so the estimate is 6. This is a good deal easier, particularly as the two values whose mean is required are often the same, as for example with 8976 ÷ 3654 when they are both 2. They also often differ only by 1, when for the mean we simply take the larger of them.

The analysis of this is in Program 8.

```
PROGRAM  8 RANDOMISE 7

   MEAN   -3.499
   Standard Deviation 10.431874

     10  13 11211614  0   0   Group
  0   15  26 27540 14  0   0 Frequ.
   1   13 48 28517  2   0

500-                          No.of trials
                                   1000

   ..........0..... ..... .....
    -40-25-10 5   20 35 50 Midpts
  -50 -35-20-5   10 25 40 55ranges
  -45-30-15 0   15 30 45
    Most positive %
29   7350 1299 5.6581986   7.3
29   2017 1299 1.5527329   2
27   1613 1282 1.2805304   1.6
27   4168 2647 1.5746128   2
27   9500 1272 7.4685535   9.5
27   7480 1289 5.8052754   7.4
26   4278 2848 0.375       3
25   3188 1578 1.8223161   2.1
25   3355 1870 5.5550857   7
24   1500 1235 1.2955466   1.6
    Most negative %
-43  1297 7353 0.17639059  0.1
-42  9271 5366 1.7277302   1
-42  7739 4480 1.7352018   1
-41  1592 9428 0.16885872  0.1
-40  9842 5679 1.6802254   1
-40  7527 4531 1.6539222   1
-39  7262 4367 1.6417399   1
-39  5800 3532 1.6421291   1
-39  1395 9664 0.16496018  0.1
-39  7320 4430 1.6502232   1
```

Program 8

Method of Estimation being tested

When the divisor is close-to it is rounded to the nearest 1 significant figure number and the quotient found by dividing to 2 significant figures (rounded down) as in Programs 6 & 7. e g. 1793 ÷ 2897. 1793 ÷ 3000 = ·59. If the divisor is not close- to, we first round the divisor up to the 1 significant figure number above and divide as far as 1 figure, rounded down. We then round the division down to the 1 significant figure below and divide as far as 1 figure, rounding down again (in Program 5 we rounded up at this point). The estimate is the mean of these, rounded up. e.g. 7263 ÷ 2597. 7263 ÷ 3000 = 2 ; 7263 ÷ 2000 = 3. Estimate 3. (rounded up).

Result : 95% of the estimates tested had errors between - 34% and + 19%.

Clearly with a mean of - 3·5%, a standard deviation of 10·4, and 95% of the 3500 estimates having errors between - 34% and + 19%, the results are not nearly as good as with Programs 6 and 7. In fact they are little better than those we obtained with Program 11 (mean - 4·1, s.d. 12, 95% range: - 24% to + 27%) when we always took the divisor to the nearest 1 significant figure number and gave the answer to 1 significant figure, or to 2 when the first figure in the estimate was a 1 or a 2 or when the divisor was a power of 10.

The results are not as good as for Program 4, where the division was always taken to 2 significant figures.

On balance therefore it would seem that although the more complicated alternatives may have been worth considering, the improvement in the results does not justify the additional trouble. It is better to try to teach the students to approximate correct to 1 significant figure and if possible to divide as far as 2 significant figures.

3. Positioning the decimal point

In each of the above cases it will of course be necessary to insert the correct number of zeros, or a decimal point. Needless to say this is where the most serious errors are most likely to occur. The reasons why we have not discussed this before are

(1) the computer programs would have been too complicated if we had included other than 4-figure numbers.

(2) the accuracy of the results are unlikely to be seriously affected by the number of figures (although there are cases where they would be somewhat affected).

(3) the teaching of this part of the process is not going to be unduly affected by the choice of method for finding the first figures of the estimate. (Here again however some methods would result in further difficulties at this stage and such methods might have to be rejected on these grounds.)

For these reasons then this aspect of estimating has been excluded from the discussion in this paper, but this should not in any way be taken to imply a lack of appreciation of its importance or the difficulties inherent in teaching it.

4. Relevant Publications

Due to the exigencies of working in the Third World, it has not been possible to make a proper survey of all the relevant literature. Aaron Buchanan's "Estimation as an essential mathematical skill" (6) however provides the background as it was in 1978. He sets out the case for teaching estimation but regrets that it "continues to slip through the structural cracks of school mathematics."

"Instruction in estimation", he says, "is not something that has been tried and failed; it has not been tried on any sustained or systematic basis."

Since 1978 a number of articles have been written about it and it is possible that individual teachers have developed their own techniques for it, but there does not yet appear to have been much in the way of "sustained and systematic" development.

Buchanan emphasizes the need for a long range strategy extending over several years in the elementary school. He admits that this would require a substantial investment of teaching time but he suggests that "In the long run, estimation may be able to play a significant role in raising the general level of quantitative literacy and mathematical understanding among students and adults": what I would call "number sense".

Buchanan's paper shows how estimation can play its part in teaching mathematical structure, measurement and problem solving. Some reference is made to the significance of the hand-held calculator, although he does not appear to go as far as the present writer who would suggest that the future of arithmetic lies in the combination of the calculator with mental estimation skills, while most of the paper and pencil algorithms of today will be gradually relegated to a course on the history of mathematics.

Buchanan makes few comments on the different techniques for estimating but one notes the same concern for the concepts of "more", "less", "between", "nearer" that have been stressed in this paper. He gives as an example the result that 346 + 562 lies between 800

and 1000, a special case of the more general rule for estimating totals given above under "Addition".

Another substantial article is by Reys and Bestgen, "Teaching and Assessing Computational Estimation Skills" (7). Data from the Second National Assessment of Mathematics (8) are used to show that performance in estimation "was more than fifteen percentage points lower "than on a corresponding exercise in which students were asked to compute the exact difference between two numbers. The example given however, determining whether 6.058 - 4.875 was closest to 1.000, 2.000, 3.000 or 9.000 illustrates the difficulty of defining what is "estimation". One could argue that a sufficient estimate for this question was 2: going to the next stage in estimating, and then deciding whether it is closer to 1 or 2 involves a lot more than this. The fact that more than 50% of 17 year-olds chose either 10.000 or 100.000 as the sum of 295.0 + 865.2 + 1.583 is perhaps a clearer example, but these could represent simply a difficulty in understanding decimals.

The Missouri Basic Essential Skills Test is quoted as stating that estimation is viewed as a basic mathematical skill but that the responsibility for checking student performance rests with the local school. Perhaps the Missouri test should be renamed. "Test of <u>some</u> Basic Essential Skills". The importance of this problem of assessment and the need to develop appropriate tests cannot be stressed too strongly.

Reys and Bestgen describe various problems associated with testing estimation skills. Multi-choice and open-ended format may test different skills. The time allowed for each question is critical. Putting the question in the context of an application may make a substantial difference.

I have myself discovered many of these problems. The "simple" assessment test used in the Course held in March 1983 (4) proved to be useless while an earlier one that seemed relatively successful involved a complicated formula for penalising those who worked too slowly (9). My current position is that we must first teach specific techniques and can then count answers correct only if they

are obtained by these methods. Perhaps when more of this has been done we may be able to develop more generally applicable tests.

The authors emphasize the need to teach "front-end" strategies for mental addition and subtraction, a point I have also made above (in 2.2). The traditionally taught "back-end" method is appropriate for pencil and paper but is positively harmful for anyone learning to estimate.

On the subject of rounding, Reys and Bestgen properly stress the complications involved in first determining whether to round up or down and then applying the appropriate process to the result. It is in any case not an appropriate method for addition or subtraction unless all numbers have the same number of figures.

In another article by the same authors, "Teaching Kids to Estimate" (10) most of the techniques suggested were versions of those discussed in this paper. A "compatible numbers strategy" would use 2400 ÷ 60 as an estimate for 2875 ÷ 66. This is indeed what many of us do ourselves, but I think it should only come after much practice with more basic methods. It requires knowing that in division both numbers can be increased or both decreased whereas in multiplying this would compound the error. The article is largely concerned with motivation and several numerical errors appear to have crept in.

"Let's Teach Mental Algorithms for Addition and Subtraction" by Gary L Musser (11) demonstrates methods for "front-end" addition and subtraction, including the use of cubes for units and "longs" for tens.

"Making Averaging Easier" by Michael S Smith (12) suggests that an average should be found by halving the numbers first and then adding them. This becomes quite complicated when an odd number has to be halved. As a method for estimating, the procedure suggested above (in 2.3) would seem to be preferable even though that too has to face the problems involved in halving odd numbers.

James Levin's "Estimation Techniques for Arithmetic" (13) recommends a process of "serial fractionalisation" which depends on visualising the positions of decimals on the number line from 0 to 1 and expressing other numbers as products of these with powers of 10. It may well be appropriate for some people, but not I think for those with whom this paper is primarily concerned.

Levin's paper does however contain a useful analysis of the need for estimating procedures. He is convinced that paper and pencil techniques are of little use in today's business or non-business worlds. He points out that the right-to-left or "back-end" algorithms have only been in use for the past four centuries and are of little value as an introduction to higher mathematics. Finally he too refers to the need for number-sense when he says "A common complaint among College teachers is that students don't seem to have a 'feel' for numbers and their computation."

Adult numeracy teaching in the "northern" world would seem an obvious place for the inclusion of estimation skills. Indeed, David Stringer's report on the television series "Make it Count" (14) states that "every programme had as one of its components at least one session on estimation". This however did not appear to include computational estimation, many of the items taking the form of "a simple game where the viewer was invited to guess how many objects were shown him". Neither did any other adult numeracy course I was able to study appear to include much in the way of computational estimation.

The agency principally concerned with adult numeracy in England is ALBSU, the Adult Literacy and Basic Skills Unit. An article by John Willis in their Newsletter (15) admits that the actual expectations of those who approach a "drop-in" centre bear little relationship to what the staff would see as their "needs". He concludes however that "we need to think more about steps in numeracy, about actual ways of teaching skills, and specifically spend more time on: manipulating numbers, on estimating, having a "feel" for the answer, knowing what sum to do when presented with a problem". Like the present writer Willis is clearly facing a real problem in determining what to teach

next and how to teach it. It may be that this sort of cry for help will result in the working out of some practical teaching programmes.

References to the need for estimation teaching are everywhere: workable proposals seem to be in short supply.

5 Conclusions

Our criteria for choosing a method of estimation for use, as distinct from concept development, are:

(1) the method should be simple enough to be taught to those whose skills do not go far beyond the learning of the multiplication tables,

(2) it should normally be possible to obtain the estimate without any working on paper,

(3) given these restrictions the estimates obtained should be as accurate as possible.

The best of the methods discussed in this paper would seem to be as follows.

5.1 For Addition

Add the thousands (or whatever the smallest units are for which the total can be "written down directly", ie writing the first figure first and the last, last) to give the least the total can be. Add 1 thousand for each number that is being added, to give the most the total can be. If an estimate is required, take the mean of this possible range. Example: 3219 + 476 + 8328 + 5654. Possible range 16 000 - 20 000. Estimate 18 000.

5.2 For subtraction

Subtract the thousands (or whatever the smallest units are for which the difference can be written down direcly) to give the estimate. The possible range is from one thousand below to one thousand above this estimate. Example: 55265 - 49832. Estimate 6000, possible range 5000 - 7000.

5.3 For the mean of two numbers

Find the difference as above, halve it and add it to the smaller number. If this involves finding half of an odd number the result is given as a possible range.

Exampel (1) 4286 and 3094. Difference 1200, so adding 600 to 3094 gives 3694 as the estimate.

(2) C837 and 1952. Difference 5000, so adding 2000 or 3000 to 1952 the possible range is 3952 - 4952.

5.4 For multiplication

If one number is "close-to", ie if its second figure is a 7, 8, 9, 0, 1 or 2, then round it to the nearest 1 significant figure number. Multiply the first figure of the other number by this and add to it whatever would be "carried" from multiplying the second number by it.

When however this involves multiplying by a power of 10, the estimate is obtained by multiplying the other number by this exactly.

When neither number is "close-to", the first figures of each number are multiplied together and their mean (rounded up) is added.

5.5 For division

Round the divisor to the nearest 1 significant figure number. Divide by it, finding at least the first figure of the quotient and the second figure as well whenever possible. Remember: the bigger the divisor the smaller the answer, and vice versa. This will enable better estimates to be obtained as experience grows.

6 Notes on the Micro-computer Program written by John Edwards for his Sinclair ZX Spectrum

The availability of only 16K memory at the time this program was written meant that techniques to economise in length were essential. The program as reproduced below would provide 500 trials of each estimating process and would take about 4 hours to run. By removing the REM statements it could accommodate 1000 trials of each and took about 6 hours to obta[illegible] the printouts shown in this article.

The use of only 4-figure numbers is explained above (5.3). Some attempt is however made to recognise the fact that many problems will in fact be expr -sed in 2 or 3-figure numbers, as such expressions could influence the accuracy of the estimates: when the randomising process produced 4-figure numbers beginning with one or more zeros therefore, these zeros were transferred to the end of the number. This increased the proportion of numbers equivalent to 2 or3-figure numbers over and above those occurring naturally in the random sequence of digits.

The randomising process on the Spectrum enables a choice to be made whereby the same series of random numbers is reproduced again or whether a different series is produced. In this experiment the same series was used for each program in any one computer run, as with all those reproduced here, but other series were used for other runs so that the total 2000 or more number-pairs used with each program in the whole experiment would in general be all different. The reference "RANDOMISE 7" at the top of each printout reproduced indicates which random number series was being used.

Finally it must be emphasised: this was a <u>working</u> program. It grew to meet each new need as it was presented, generally within a few hours. It could now of course be re-written in polished form but then it would no longer illustrate the flexibility and practical usefulness of having a micro-computer, and its programmer, in the home.

```
  1 REM ■ © J.M.EDWARDS 1983
  2 REM ■ SPECTRUM 1 16K
  3 REM ■ ALLENS APPROXIMATIONS
  4 GO TO 140
  6 LET EX=VAL A$*VAL B$: RETUR
N
  7 LET EX=VAL A$/VAL B$: RETUR
N
  9 REM ■ PROGRAM 1
 10 GO SUB 6: LET AP=1E6*FN Q(A
$)*FN Q(B$)
 12 RETURN
 19 REM ■ PROGRAM 2
 20 GO SUB 30
 22 IF NOT C THEN LET AP=(VAL A
$(1)*VAL B$(1)+(VAL A$(1)+1)*(VA
L B$(1)+1))/2*1E6: RETURN
 24 RETURN
 29 REM ■ PROGRAM 3
 30 GO SUB 6: LET C=FN R(A$)
 32 IF NOT C THEN LET C$=A$: LE
T A$=B$: LET B$=C$: LET C=FN R(A
$)
 34 IF NOT C THEN LET AP=(VAL A
$(1)*(VAL B$(1)+INT ((VAL A$(1)+V
AL B$(1))/2+.5))*1E6: RETURN
 36 LET C$=STR$ (FN Q(A$)*VAL B
$(2)): LET AP=(FN Q(A$)*VAL B$(1
)+(VAL C$(1) AND LEN C$>1))*1E6
 38 RETURN
 39 REM ■ PROGRAM 4
 40 GO SUB 7: LET C$=STR$ FN Q(
B$)
 42 LET D$=STR$ (VAL A$/VAL C$)
 44 LET AP=VAL D$( TO 2)
 45 REM ■ FACTORS OF 10 FIDDLE
 46 IF ABS ((AP-EX)/EX)>1 THEN
LET AP=AP/10: GO TO 45
 48 RETURN
 49 REM ■ PROGRAM 5
 50 GO SUB 7: LET C$=STR$ (VAL
A$/(VAL B$(1)+1)): LET C$=FN D$(
C$)
 52 LET D$=STR$ INT (VAL A$/VAL
 B$(1)): LET D$=FN E$(D$)
 54 IF VAL C$>VAL D$ THEN LET D
$=D$+"0": GO TO 54
 56 LET AP=(VAL C$+VAL D$)/2: G
O TO 46
 59 REM ■ PROGRAM 6
 60 GO SUB 7: LET C=FN R(B$)
 62 IF C THEN LET C$=STR$ (VAL
A$/FN Q(B$)): LET AP=VAL C$( TO
2): GO TO 46
 64 GO TO K+2-(12 AND K=50)
 65 REM ■ PROGRAM 7
 70 GO TO 60
 72 LET C$=STR$ (VAL A$/VAL B$(
1))
 74 LET D$=STR$ (VAL C$( TO 2)/
VAL B$(1))
 76 LET AP=VAL C$( TO 2)-INT (V
AL D$/2): GO TO 46
```

```
  79 REM ■ PROGRAM 8
  80 GO TO 60
  82 LET C$=STR$ (VAL A$/(VAL B$
(1)+1))
  84 IF A$(1)=B$(1) THEN LET AP=
INT (5+VAL C$(1)/2): GO TO 46
  86 LET D$=STR$ (VAL A$/VAL B$(
1))
  88 LET AP=INT ((VAL D$(1)+VAL
C$(1))/2+.9): GO TO 46
  89 REM ■ PROGRAM 9
  90 GO SUB 40: LET AP=VAL D$(1)
  92 IF CODE C$=49 THEN LET AP=V
AL A$
  94 GO TO 46
  99 REM ■ PROGRAM 10
 100 GO SUB 40: LET AP=VAL FN D$
(D$): IF CODE C$=49 THEN LET AP=
VAL A$
 102 GO TO 46
 109 REM ■ PROGRAM 11
 110 GO SUB 40: LET AP=VAL D$( T
O 1+(CODE D$(51)): IF CODE C$=49
 THEN LET AP=VAL A$
 112 GO TO 46
 119 REM ■ PROGRAM 12
 120 GO SUB 50: IF CODE B$=57 TH
EN LET C$=A$( TO 2)
 122 IF CODE B$=49 THEN LET D$=S
TR$ (VAL A$( TO 2)+1)
 124 GO TO 54
 129 REM ■ PROGRAM 13
 130 GO SUB 30: IF C=1 OR C=10 T
HEN LET AP=VAL B$*1E3*C
 132 RETURN
 139 REM ■ WORK STARTS HERE
 140 FOR K=10 TO 130 STEP 10
 150 GO TO 1010
 160 LPRINT ''"PROGRAM ";K/10;"
RANDOMISE 4"''
 170 PRINT AT 10,4;"WORKING OUT
";MAX;" SUMS"''"FOR PROG.";K/10
 180 FOR J=1 TO MAX
 189 REM ■ CHOOSE 2 NUMBERS
 190 LET A$=FN A$(): LET B$=FN A
$()
 200 REM ■ EXACT+APPROX
 210 GO SUB K
 228 REM ■ P$:% 0=132,+is more
 232 LET P$=CHR$ (132+(AP-EX)*10
0/EX)
 239 REM ■ TEST LINE ; VARIES
 240 REM IF C$="#" THEN PRINT J;
" ";A$;" ";B$;" ";C$;" ";" ";TAB
 19;AP;" ";TAB 29; PAPER 6;CODE
P$-132
 241 REM ■ BINARY SORT
 242 LET L=0: LET U=J
 244 LET F=INT ((U+L+1)/2)
 250 IF S$(F)=P$ THEN GO TO 260
 254 IF S$(F)>P$ THEN LET L=F: G
O TO 260
 258 LET U=F
 260 IF (U-L)>1 THEN GO TO 244
 270 LET F=U
 279 REM ■ RECORD %
 280 LET S$=S$( TO F-1)+P$+S$(F
TO )
 300 REM ■ TOP & BOTTOM
 310 IF F>LIM AND (J-F)>LIM THEN
 GO TO 540
 400 REM ■ SORT TOP
 410 FOR G=1 TO LEN H$ STEP 9
 420 IF H$(G)>P$ THEN NEXT G
 430 LET H$=H$( TO G-1)+P$+A$+B$
+H$(G TO )
 500 REM ■ SORT BOTTOM
 510 FOR G=1 TO LEN H$ STEP 9
 520 IF L$(G)<P$ THEN NEXT G
 530 LET L$=L$( TO G-1)+P$+A$+B$
+L$(G TO )
 540 NEXT J
 590 GO TO 700
 600 REM ■ PRINT TOP AND BOTTOM
 605 CLS
 610 LET R$=H$+L$
 625 PRINT TAB 4;"Most positive
%"
 630 FOR G=1 TO LEN R$ STEP 9
 645 LET A$=R$(G+1 TO G+4): LET
 B$(G+5 TO G+8)
  3 PRINT CODE R$(G)-132;TAB 4;
 " ";B$;" ";TAB 14;:      SUB K:
RINT EX;TAB 23+2*(K)      OP
```

```
 647 IF G=LEN H$-8 THEN PRINT TA
B 4;"Most negative %"
 650 NEXT G
 660 COPY
 670 CLS
 680 NEXT K
 690 STOP
 700 REM ■ CLASS -2 TO 2 ETC.
 701 LET G=5: DIM H(24)
 702 PRINT AT 10,10;"CLASSING ";
MAX;" RESULTS"
 705 FOR F=1 TO LEN S$
 710 IF CODE S$(F)>(189-G) THEN
NEXT F
 730 LET G=G+5
 740 LET H(G/5)=F
 745 IF F>=LEN S$ THEN LET F=F-1
: GO TO 745
 750 IF G<120 THEN GO TO 710
 760 LET H(24)=LEN S$
 762 LET H(1)=1
 764 CLS
 770 FOR F=24 TO 2 STEP -1
 780 LET H(F)=H(F)-H(F-1)
 790 NEXT F
 799 REM ■ DRAW IT
 800 FOR F=2 TO 23
 805 PRINT AT 19+FN F(F),F+(F>=1
2);5*(F-12)
 806 PRINT AT 4+FN F(F),F+1;H(25
-F)
 810 FOR G=11 TO 14: PLOT 8*F+G,
25
 820 DRAW 0,H(25-F)*150/MAX
 830 NEXT G: NEXT F
 840 PRINT AT 9,0;INT (max/2);"-
"
 844 PRINT AT 4,26;"Group";AT 5,
26;"Frequ."
 850 PRINT AT 9,18;"No.of trials
";TAB 23;MAX
 860 PRINT AT 19,26;"Midpts";AT
20,26;"ranges"
 900 REM ■ MEAN + S.DEVIATION
 910 LET P=0
 920 FOR F=1 TO MAX
 930 LET P=P+CODE S$(F)
 940 NEXT F
 950 LET P=P/MAX
 951 REM ■ P=MEAN,132=0
 960 LET SS=0
 970 FOR F=1 TO MAX
 980 LET SS=SS+(CODE S$(F)-P)*(C
ODE S$(F)-P)
 990 NEXT F
 999 PRINT AT 1,0;TAB 3;"MEAN  "
;P-132'TAB 3;"Standard Deviation
";SQR (ss/max)
1000 COPY : GO TO 600
1010 RANDOMIZE 4
1024 REM ■ TOP+BOTTOM
1025 LET LIM=10
1026 LET MAX=500
1029 REM ■ S$ STORES % ERRORS
1030 DIM S$(MAX)
1039 REM ■ A$ etc.MUST BE 4-FIG
1040 DIM A$(4): DIM B$(4)
1049 REM ■ H$,L$ KEEP MAX/MIN
1050 DIM H$(9*LIM)
1060 DIM L$(9*LIM)
1069 REM ■ L$ MUST START HIGH
1070 FOR F=1 TO LEN L$ STEP 9: L
ET L$(F)=CHR$ 255: NEXT F
1089 REM ■ 3-LINE DISPLAY FN
1090 DEF FN F(F)=(F/3=INT (F/3))
*2+((F+1)/3=INT ((F+1)/3))
1091 REM ■ FN Q:1st.SIG.FIGTOR
1092 DEF FN Q(A$)=INT (VAL A$/10
00+.5)
1093 REM ■ FN R IS 0 OR Q IFF
1094 DEF FN R(A$)=FN Q(A$) AND (
A$(2)>"6" OR A$(2)<"3")
1099 REM ■ 4-FIG RANDOM NO.
1100 DEF FN A$()=STR$ INT (RND*1
E4)+"000"
1109 REM ■ 2 FIGS IFF 1ST.IS"1"
1110 DEF FN D$(D$)=D$( TO 1+(D$(
1)="1"))
1119 REM ■ ROUNDS UP
1120 DEF FN E$(D$)=STR$ (VAL FN
D$(D$)+1)
1140 GO TO 160
9999 SAVE "allen"
```

38

7. Acknowledgements

The encouragement and help of all those who have supported the work of the Numeracy Project in Papua New Guinea is warmly acknowledged, particularly those who have made it possible for the author to complete and record this experiment while on leave in England. John Edwards did all the work on the computer, Carol Tipson and Vera Forster did the typing, Graham Hall and the Elm Bank Teachers Centre of the Coventry Local Education Authority generously provided the first set of photocopies. Martha Spieth and Aaron Buchanan helped with the brief literature search in the United States. Also Mary Edwards whose well-earned holiday was, not for the first time, disrupted.

REFERENCES

(1) Edwards, A THE HAND HEID CALCULATOR IN A THIRD WORLD COUNTRY. The Numeracy Project of Papua New Guinea, 1982. This is a slightly reduced version of the doctorial thesis presented by the author in 1982 with the assistance of Professor Bajpai of Loughborough University. (ERIC reference: Ed 222 381)

(2) Cockcroft, W H MATHEMATICS COUNTS. REPORT OF INQUIRY INTO THE TEACHING OF MATHEMATICS IN SCHOOLS UNDER THE CHAIRMANSHIP OF DR W H COCKCROFT. Her Majesty's Stationery Office, 1982.

(3) Hayter, J EVALUATION STUDY OF THE PAPUA NEW GUINEA HIGH SCHOOL MATHEMATICS CURRICULUM. An internal report of the Department of Education (Draft) 1980.

(4) Edwards, A CALCULATE AND ESTIMATE. A REPORT ON THE BACKGROUND AND CONDUCT OF A COURSE HEID AT THE ADMINISTRATIVE COLLEGE OF PAPUA NEW GUINEA, MARCH 7-11, 1983. The Numeracy Project of Papua New Guinea, 1983.

(5) Edwards, A UNDERSTANDING STATISTICS. To be published in 1983 by the Numeracy Project of the Administrative College of Papua New Guinea. Currently only in draft form.

(6) Buchanan, Aaron D ESTIMATION AS AN ESSENTIAL MATHEMATICAL SKILL. Professional Paper 39 of the Southwest Regional Laboratory for Educational Research and Development, Los Alamitos, Calif, 1978.

(7) Reys, R E and Bestgen, B J TEACHING AND ASSESSING COMPUTATIONAL ESTIMATION SKILLS. The Elementary School Journal v.82, n.2, November 1981.

(8) Carpenter T P; Corbitt M K; Kepner H S; Lindquist M M and Reys R E RESULTS AND IMPLICATIONS OF THE SECOND NAEP MATHEMATICS ASSESSMENT - SECONDARY SCHOOL. Mathematics Teacher 73, p 329-38. May 1980.

(9) Edwards A AN EXPERIMENT IN TEACHING AND ASSESSING CALCULATOR SKILLS AND, MORE PARTICULARLY, MENTAL NUMBER SKILLS THAT REMAIN RELEVANT WHEN A CALCULATOR IS AVAILABLE. Numeracy Project of Papua New Guinea, July 1982.

(10) Bestgen B & Reys R TEACHING KIDS TO ESTIMATE. Instructor, April 1982.

(11) Musser G L LET'S TEACH MENTAL ALGORITHMS FOR ADDITION AND SUBTRACTION. Arithmetic Teacher v.29, n.8, p40-42, April 1982.

(12) Smith M S MAKING AVERAGING EASIER. Arithmetic Teacher v 29, n 4, p40-44, Dec 1981.

(13) Levin J A ESTIMATION TECHNIQUES FOR ARITHMETIC: EVERYDAY MATHS AND MATHEMATICS INSTRUCTION. Educational Studies in Mathematics v 12 (1981) pp 421-434.

(14) Stringer D MAKE IT COUNT: A REPORT. Independent Broadcasting Authority 1979.

(15) Willis J NUMERACY: NEEDS AND DEMANDS IN A DROP-IN CENTRE. In Newsletter No.11 of the Adult Literacy and Basic Skills Unit, London, Feb/March 1983.

ANR 23